MTH 114
LOGIC AND LINEAR ALGEBRA
LECTURE NOTES

for ND

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COURSE OUTLINE:

1. Logic and abstract thinking
2. Permutation and Combination
3. Binomial Theorem of Algebraic Expressions
4. Matrices and Determinants

Contents

COURSE OUTLINE: **i**

1 Logic and abstract thinking **4**

1.1 Introduction 4

1.1.1 Logic 4

1.1.2 Linear Algebra 4

2 Logic **5**

2.1 Basic Concepts 5

2.1.1 A (Simple) Statement 5

2.1.2 Non - logical Statement 5

2.1.3 Truth Value of a Statement 6

2.1.4 Connectives 6

2.1.5 Compound Statement 7

2.2 Conversion of Statement to Symbols Form 7

2.3 Converse, Inverse and Contrapositive 9

2.3.1 Converse of a Conditional Statement 9

2.3.2 Inverse of a Conditional Statement 9

2.3.3 Contrapositive 9

2.4 Parenthesis 10

2.5 Truth Table 11

2.5.1 Negation 12

2.5.2 Conjunction 12

2.5.3 Disjunction 12

2.5.4 Implication 12

2.5.5 Bi-implication 12

2.6 Tautology and Contradiction 13

2.6.1 Tautology 13
2.6.2 Contradiction 14
2.7 Logical Equivalence 14
2.8 Quantifiers 16
2.8.1 Predicate 16
2.8.2 Quantification 16
2.8.3 Universal Quantifier 16
2.8.4 Existential Quantifiers 17
2.9 Statement Translation Using Quantifiers 18
2.10 The Scope of a Quantifier 19
2.11 Bound and Free Variables 20
2.11.1 Bound Variable 20
2.11.2 Free Variable 20
2.12 Formula 21
2.12.1 Argument 21
2.12.2 Validity of an Argument 21

3 Permutation and Combination 23
3.1 Permutation 23
3.1.1 Permutation Formula 24
3.1.2 Permutation of Repeated Objects 27
3.1.3 Permutation with Restrictions 28
3.1.4 Circular Permutation 28
3.2 Combination 29

4 Binomial Theorem of Algebraic Expressions 32
4.1 Mathematical Induction 32
4.2 The Pascal Triangle 33
4.3 Binomial Theorem for Positive Integer Indices 36
4.3.1 Finding the Constant term (independent of x) in an Expansion 40
4.3.2 Finding the Coefficient of a Particular term in an Expansion 42
4.4 Binomial Theorem for Negative and Fractional Indices 43
4.5 Application of Binomial Theorem to Approximation . . 45

5 Matrices and Determinants 47

5.1 Matrices 47

5.1.1 Order of a Matrix 47

5.1.2 Types of Matrices 48

5.2 Addition/Subtraction of Matrices 51

5.3 Multiplication of Matrices 52

5.3.1 Scalar Multiplication 52

5.3.2 Matrix Multiplication 52

Chapter 1

Logic and abstract thinking

1.1 Introduction

1.1.1 Logic

Logic is the study of formal reasoning based upon statements or propositions. The basic principle of logic is centered on 2 laws; the law of contradiction which states that a statement cannot be both true and false, and the law of excluded middle which stresses that a statement must be either true or false. A simple example of how logic works is as follows; consider the statement,

"Umar hates all black objects"

in a logical sense, this statement can be written as:

"In all objects, if an object is black then Umar hates it"

Symbolically, we write:

For all x, if x is black, then Umar hates x

1.1.2 Linear Algebra

This is the branch of Mathematics that deals with the theory of systems of linear equations, matrices, vector spaces, determinants and linear transformations.

Chapter 2

Logic

2.1 Basic Concepts

In this section, the basic concepts to be used are explained

2.1.1 A (Simple) Statement

In logic, a statement (or a Proposition) is a meaningful declarative sentence that is either True (T) or False (F). Examples of logical statements include the following:

1. "I am in ND 1"
2. "The phone is ringing"
3. "$3 + 5 = 30$"
4. "The colour of the pen is red"
5. "I am eating"
6. "$\frac{6}{3} = 2$" etc.

2.1.2 Non - logical Statement

These are statements that can niether be true or false. Examples include the following:

1. "How are you?"
2. "Is he a boy?"

3. "3 + 5"
4. "Welldone!"
5. "Who are you"
6. "$\frac{6}{3}$" etc.

All questions and exclamations are non logical statements since we cannot say wether they are True or False.

2.1.3 Truth Value of a Statement

The truth value of a statement is the state in which the statement is, either True (T) or False (F). If the statement is True, then its truth value is **T** and if it is false, then its truth value is **F**.
Statements are usually represented by symbols (or letters); for example, the statement

"The sky is black"

can be represented by **p**. We write

$$\mathbf{p} := \textit{The sky is black}$$

Thus, the truth value of the statement **p** is **F** since the sky is not black.

2.1.4 Connectives

A logical connective (also called a logical operator) is a symbol or word used together with a statement or to connect 2 or more statements. The commonly used connectives (essential connectives) include the following:

S/N	Name	Word Used	Symbol
1	Negation	NOT	˜
2	Conjunction	AND	$\cap$
3	Disjunction	OR	$\cup$
4	Implication	IF...THEN...	$\longrightarrow$
5	Bi-implication	...IFF...	$\longleftrightarrow$

The following examples illustrates how statement(s) can be used together with connectives. Consider the two statements:

Umar likes reading
Umar hates traveling

If we let **p**:= *Umar likes reading*, **q**:= *Umar hates traveling*, then

$$\mathbf{p} \cap \mathbf{q} := \textit{Umar likes reading } \textbf{AND} \textit{ hates traveling}$$

2.1.5 Compound Statement

A compound statement is a statement consisting of combination of statement(s) and connective(s). For example:

1. If **p** := The colour of the car is red, then,
   ˜**p** := The colour of the car is **NOT** red.
2. If **p** := $2 + 3 = 5$, **q** := $\frac{7}{3} = 4$, then,
   $\mathbf{p} \cap \mathbf{q}$:= $2 + 3 = 5$ **AND** $\frac{7}{3} = 4$
3. If **p** := All Nigerians are tall, **q** := All Nigerians are happy, then,
   $\mathbf{p} \cup \mathbf{q}$:= All Nigerians are tall **OR** happy.
4. If **p** := x is a prime number, **q** := x is a factor of 24, **r** := x is a prime factor of 24, then,
   IF $(\mathbf{p} \cap \mathbf{q})$ **THEN** $\mathbf{r} = (\mathbf{p} \cap \mathbf{q}) \longrightarrow \mathbf{r}$:= If x is a prime number **AND** a factor of 24, **THEN** x is a prime factor of 24.
5. If **p** := He is smiling, **q** := He is happy, then,
   p IFF q = $\mathbf{p} \longleftrightarrow \mathbf{q} := (\mathbf{p} \longrightarrow \mathbf{q})$ **AND** $(\mathbf{q} \longrightarrow \mathbf{p})$:=
   IF He is smiling, **THEN** He is happy **AND IF** He is happy, **THEN** He is smiling

2.2 Conversion of Statement to Symbols Form

The following table illustrates how to convert statements to symbols:

STATEMENT	ASSUMPTION	SYMBOLS	MEANING
NEGATION ˜			
Emeka is not tall	**p** := Emeka is tall	˜**p**	Emeka is not tall
She is not ugly	**q** := She is ugly	˜**q**	She is not ugly
CONJUNCTION $\cap$			
41 is prime	**p** := 41 is prime	**p** $\cap$ ˜**q**	41 is prime
41 is not even	**q** := 41 is even		but not even
DISJUNCTION $\cup$			
He is at home	**p** := He is at home	**p** $\cup$ **q**	He is at home
He is at work	**q** := He is at work		or work
IMPLICATION $\longrightarrow$			
2 can divide 8	**p** := 2 can divide 8	**p** $\longrightarrow$ **q**	If 2 can divide 8
8 is even	**q** := 8 is even		then 8 is even

Examples: Convert the following statements to symbol form using appropriate assumptions

1. The sun is hot but it is not humid.
2. If John doesn't pass then he will loose his scholarship and drop out of school.
3. If it rains and you don't open your umbrella then you will get wet.
4. If your car won't start or you don't wake up on time then you will miss your interview and you will not get the job.
5. If you elect Hillary then Hillary will make sure that the budget is not padded, corruption will cease and there will be N5,000 social benefit.
6. If the cake gets hot then the icing melts and if the icing melts then the cake cannot be used at the wedding reception.
7. If the super eagles win the world cup or Kano pillars win the National cup then Nigerians will be overjoyed and dance in the streets.

Solution:

2.3 Converse, Inverse and Contrapositive

The converse, inverse and contrapositive of a conditional statement $\mathbf{p} \longrightarrow \mathbf{q}$ can be found as follows:
Note: In the statement $\mathbf{p} \longrightarrow \mathbf{q}$, $\mathbf{p}$ is called the *hypothesis* of the statement while $\mathbf{q}$ is called the conclusion of the statement.

2.3.1 Converse of a Conditional Statement

The converse of a conditional statement, $\mathbf{p} \longrightarrow \mathbf{q}$, is a statement obtained by interchanging the positions of the hypothesis and the conclusion i.e.
The converse of $\mathbf{p} \longrightarrow \mathbf{q}$ is $\mathbf{q} \longrightarrow \mathbf{p}$

2.3.2 Inverse of a Conditional Statement

The inverse of a conditional statement, $\mathbf{p} \longrightarrow \mathbf{q}$, is a statement obtained by negating both the hypothesis and the conclusion without changing their positions i.e.
The inverse of $\mathbf{p} \longrightarrow \mathbf{q}$ is $\tilde{}\mathbf{p} \longrightarrow \tilde{}\mathbf{q}$

2.3.3 Contrapositive

The contrapositive of a conditional statement, $\mathbf{p} \longrightarrow \mathbf{q}$, is a statement obtained by finding both the converse and inverse of the statement i.e. interchanging the positions of both hypothesis and conclusion and also taking the negation of both, thus we have
The contrapositive of $\mathbf{p} \longrightarrow \mathbf{q}$ is $\tilde{}\mathbf{q} \longrightarrow \tilde{}\mathbf{p}$

Examples:

1. Consider the conditional statement

If he reads well then he will pass his examinations

The statement can be written as **p**⟶**q**, where
p:= *he reads well* and
q:= *he will pass his examinations*
Now the converse of the statement is
q⟶**p** interpreted as: If he will pass his examinations then he reads well.
The inverse of the statement is
˜**p**⟶˜**q** interpreted as: If he doesn't read well then he will not pass his examinations
and the contrapositive of the statement is
˜**q**⟶˜**p** interpreted as: If he will not pass his examinations then he doesn't read well.

2. Obtain the converse, inverse and contrapositive of the following statement
   a. If you are in ND 2 then you must go for SIWES
   b. If it rains today then your clothes will be wet
   c. If today is Wednesday then tomorrow will be Thursday
   d. If 8 is divisible by 2 then 8 is an even number **Solution:**

2.4 Parenthesis

Parenthesis i.e. ( ) are very useful when it comes to logical statements. They are used to prevent ambiguity in a statement and also used to show the order of operations. For example, consider the statement

$$\mathbf{p} \cap \mathbf{q} \cup \mathbf{r}$$

It could mean *"either* **p** *and* **q** *or* **r***"* or *"***p** *and either* **q** *or* **r***"*. This brings confusion as to which of the interpretation is correct for the statement. To prevent this confusion, grouping is done with parenthesis to get the actual meaning of the statement.
A grouped version of the statement

$$\mathbf{p} \cap \mathbf{q} \cup \mathbf{r}$$

can be

$$(\mathbf{p} \cap \mathbf{q}) \cup \mathbf{r}$$

which means *""either* **p** *and* **q** *or* **r***"*, or it can be

$$\mathbf{p} \cap (\mathbf{q} \cup \mathbf{r})$$

which means *"***p** *and either* **q** *or* **r***"*.
Parenthesis show grouping and tells you to start with the statement in the group before others. A statement may contain more than one set of parenthesis for example
$(\mathbf{p} \cap \mathbf{q}) \cup (\mathbf{q} \cap \mathbf{r})$
$\tilde{}(\mathbf{p} \cap \mathbf{q}) \cup \mathbf{r}$
$(\mathbf{p} \cup (\tilde{}\mathbf{p} \cap \mathbf{r}) \cup (\mathbf{p} \cap \mathbf{q})$
$\tilde{}(\mathbf{p} \cup \mathbf{q}) \longrightarrow \tilde{}\mathbf{p}$, etc.

2.5 Truth Table

It is known that each logical statement has a truth value which is either **T** or **F**. The truth value of a compound statement can also be found and this depends on the truth values of each component statement. Take for example the compound statement

$$\mathrm{C} = \mathbf{p} \cap \mathbf{q}$$

Suppose the truth value of **p** is **T** and that of **q** is **F**, then the truth value of the compound statement $\mathrm{C} = \mathbf{p} \cap \mathbf{q}$ will be **T** and **F** which is logically **F**.
Take another example, the compound statement

$$\mathrm{D} = \mathbf{p} \cup \mathbf{q}$$

Suppose the truth value of **p** is **T** and that of **q** is **F**, then the truth value of the compound statement $\mathrm{D} = \mathbf{p} \cup \mathbf{q}$ will be **T** or **F** which is logically **T**.
Now lets define what a truth table is. The truth table of a statement is a diagram (usually arranged in rows and columns) that shows all truth values of the statement from all possible combinations of the truth values of its component statements. The following gives the truth tables of some statements consisting of essential connectives:

2.5.1 Negation

p	**˜p**
T	F
F	T

2.5.2 Conjunction

p	**q**	$\mathbf{p \cap q}$
T	T	T
T	F	F
F	T	F
F	F	F

2.5.3 Disjunction

p	**q**	$\mathbf{p \cup q}$
T	T	T
T	F	T
F	T	T
F	F	F

2.5.4 Implication

p	**q**	$\mathbf{p \longrightarrow q}$
T	T	T
T	F	F
F	T	T
F	F	T

2.5.5 Bi-implication

Note that, the bi-implication statement can be broken down as follows $\mathbf{p} \longleftrightarrow \mathbf{q} \equiv (\mathbf{p} \longrightarrow \mathbf{q}) \cap (\mathbf{q} \longrightarrow \mathbf{p})$ Therefore, its truth table will also be broken down into its components as follows

$\mathbf{p}$	$\mathbf{q}$	$\mathbf{p} \longrightarrow \mathbf{q}$	$\mathbf{q} \longrightarrow \mathbf{p}$	$(\mathbf{p} \longrightarrow \mathbf{q}) \cap (\mathbf{q} \longrightarrow \mathbf{p})$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Examples: Draw the truth tables of the following compound statements:

1. $(\tilde{}\mathbf{p} \longrightarrow \tilde{}\mathbf{q})$
2. $\mathbf{p} \longrightarrow (\tilde{}\mathbf{q} \cup \mathbf{r})$
3. $\mathbf{p} \cap \tilde{}\mathbf{q}$
4. $(\tilde{}\mathbf{p} \cup \mathbf{q}) \longrightarrow \mathbf{p}$
5. $\mathbf{p} \longrightarrow (\mathbf{q} \cap \mathbf{p})$
6. $(\mathbf{p} \longrightarrow \tilde{}\mathbf{q}) \cup \{\mathbf{r}$
7. $(\mathbf{p} \cup \tilde{}\mathbf{q}) \longrightarrow (\mathbf{p} \cup \mathbf{r})$
8.
9.
10.

2.6 Tautology and Contradiction

2.6.1 Tautology

A compound statement is called a tautology if its truth value is always **T** no matter the truth values of its components. To know whether a statementt is a tautology, its truth table should be obtained and if all the truth values of the last column are **T**, then that statement is a tautology. For example:

Examples: Show that the following statements are tautologies

1. $\mathbf{p} \cup (\tilde{}\mathbf{p})$
2. $\mathbf{p} \longrightarrow (\mathbf{p} \cup \mathbf{q})$

3. $(\mathbf{p} \cap \mathbf{q}) \cap (\tilde{}\mathbf{q} \cap \mathbf{r})$
4.
5.

2.6.2 Contradiction

A compound statement is called a contradiction if its truth value is always **F** no matter the truth values of its components. To know whether a statement is a contradiction, its truth table should be obtained and if all the truth values of the last column are **F**, then that statement is a contradiction. For example:

Examples: Show that the following statements are contradictions

1. $\mathbf{p} \cap (\tilde{}\mathbf{p})$
2.
3.
4.
5.

2.7 Logical Equivalence

Two or more statements are equivalent if they have the same truth tables i.e. if the last column of both truth tables of the statements is the same. We write $\mathbf{p} \equiv \mathbf{q}$ to mean $\mathbf{p}$ is equivalent to $\mathbf{q}$. Therefore to know whether two statements are equivalent or not, their truth tables need to be obtained and compared.

Examples: Show that the statements in the following laws are equivalent

1. Cummutative Laws

$$\mathbf{p} \cup \mathbf{q} \equiv \mathbf{q} \cup \mathbf{p}$$

$\mathbf{p} \cap \mathbf{q} \equiv \mathbf{q} \cap \mathbf{p}$

2. Associative Laws

$(\mathbf{p} \cup \mathbf{q}) \cup \mathbf{r} \equiv \mathbf{p} \cup (\mathbf{q} \cup \mathbf{r})$

$(\mathbf{p} \cap \mathbf{q}) \cap \mathbf{r} \equiv \mathbf{p} \cap (\mathbf{q} \cap \mathbf{r})$

3. Distributive Laws

$\mathbf{p} \cup (\mathbf{q} \cap \mathbf{r}) \equiv (\mathbf{p} \cup \mathbf{q}) \cap (\mathbf{p} \cup \mathbf{r})$

$\mathbf{p} \cap (\mathbf{q} \cup \mathbf{r}) \equiv (\mathbf{p} \cap \mathbf{q}) \cup (\mathbf{p} \cap \mathbf{r})$

4. Identity

$\mathbf{p} \cup \mathbf{F} \equiv \mathbf{p}$

$\mathbf{p} \cap \mathbf{T} \equiv \mathbf{p}$

5. Complement Properties

$\mathbf{p} \cup \tilde{}\mathbf{p} \equiv \mathbf{T}$

$\mathbf{p} \cap \tilde{}\mathbf{p} \equiv \mathbf{F}$

6. Double Negation

$\mathbf{p}\ (\tilde{}\mathbf{p}) \equiv \mathbf{p}$

7. Idempotent Laws

$\mathbf{p} \cup \mathbf{p} \equiv \mathbf{p}$

$\mathbf{p} \cap \mathbf{p} \equiv \mathbf{p}$

8. Demorgan's Laws

$\tilde{}(\mathbf{p} \cup \mathbf{q}) \equiv \tilde{}\mathbf{p} \cap \tilde{}\mathbf{q}$

$\tilde{}(\mathbf{p} \cap \mathbf{q}) \equiv \tilde{}\mathbf{p} \cup \tilde{}\mathbf{q}$

9. Universal Bound Laws (Domination)

$\mathbf{p} \cup \mathbf{T} \equiv \mathbf{T}$

$\mathbf{p} \cap \mathbf{F} \equiv \mathbf{F}$

10. Absorption Laws

$\mathbf{p} \cup (\mathbf{p} \cap \mathbf{q}) \equiv \mathbf{p}$

$\mathbf{p} \cap (\mathbf{p} \cup \mathbf{q}) \equiv \mathbf{p}$

2.8 Quantifiers

2.8.1 Predicate

A predicate can be defined as a statement that contains a variable (unknown) for example: *"x is greater than 5"*
"The colour of y is white"
"x is in ND 2"
"x and y are both numbers" etc.
A Predicate with a variable x is usually represented by $P(x)$. The truth value of a predicate can only be known if the variable (unknown) in the predicate is known.
Thus, the truth value of the predicate

$$P(x) := x \text{ is less than } 5.$$

is **T** if $x = 1$ and **F** when $x = 10$.
A predicate with variable(s) is also called an *atomic formula*.

2.8.2 Quantification

Consider the predicate

$$P(x) := x \text{ is greater than } 7$$

$P(x)$ is not a proposition because we do not know its truth value but if the value of x is given, then the predicate becomes a proposition (i.e. logical statement).
Another way of converting a predicate to a proposition is by restricting the values the variable can take to a particular set; for example if x is a number between 10 and 15, then the predicate $P(x) := x$ is greater than 7 becomes a proposition whose truth vale is **T**. This second way is called *quantification*.
There are 2 types of quantifiers namely; universal and existential quantifiers.

2.8.3 Universal Quantifier

Consider the following statements,
$1 + 1 + 1 = 3 \times 1$ and

$2 + 2 + 2 = 3 \times 2$ and
$3 + 3 + 3 = 3 \times 3$ and
$4 + 4 + 4 = 3 \times 4$ and so on.
We can use a single predicate to express all the above statements. The single predicate to represent all the above statements is

$$P(x) := x + x + x = 3 \times x$$

This statement can be interpreted as
For every number x, $x+x+x = 3\times x$. Mathematically (using symbols), this is written as

$$\forall x\colon\ P(x)$$

where $\forall$ represents the phrase "for all" or "for every" or "for each" etc.
The statement

$$\forall x\colon\ P(x)$$

can be translated in English as

for each x, (the predicate) $P(x)$ holds

$\forall$ is called the universal quantifier for the predicate $P(x)$ and $\forall x$ means all the object x in the domain of discussion.

2.8.4 Existential Quantifiers

This is a quantifier which is translated as "there exists" or "there is at least one" or "for some" etc. The existential quantifier is usualy denoted by an inverted E i.e. $\exists$.
For example lets consider the following set of logical statements
$1 + 1 = 20$ or
$2 + 2 = 20$ or
$3 + 3 = 20$ or
$4 + 4 = 20$ and so on.
A predicate can be used to represent all the statements above and this predicate is,

$$P(x) := x + x = 20$$

Since we know that there is at least one number which if we put as x in the predicate, it will be satisfied and that number is 10. If $x = 10$ then the predicate $P(x) := x + x = 20$ will be true. Thus, using an existential quantifier we can have a logical statement as,

"there is a number x such that $P(x) := x + x = 20$ holds"

Mathematically (using symbols), the above statement can be written as

$$\exists x\colon\ P(x)$$

translated in English as

"there exists x such that (the predicate) $P(x)$ holds"

Note that the negation of a universal quantifier is the existential quntifier and likewise the negation of the existential quantifier is the universal quantifier.

2.9 Statement Translation Using Quantifiers

Lets consider some statements and see how they can be translated using quantifiers.

Examples: Translate the following statements to predicate logic using appropriate quantifiers and showing all your assumptions

1. "Every book has an author"
   Solution:
   Let
   B be the set of books,
   A be the set of authors and
   $P(x, y) := x$ is the author of y, then the statement "Every book has an author" can be written symbolically as

$$\forall x \in B, \exists a \in A : P(a, x)$$

2. "All humans are mortal"
   Solution:
   Let H be the set of humans and
   $P(x) := x$ is mortal, then the statement can be written symbolically as

$$\forall x \in H : P(x)$$

3. "Some students missed todays lecture"
4. "There is an author who has not written a book"
5. "If a thing is worth doing, then it is worth doing well"
6. "All that glitters is not gold"
7. "Every body except Alex did the assignment"

2.10 The Scope of a Quantifier

The scope of a quantifier is the formula directly following the quantifier. The scope is the extend of the application (effect) of the quantifier. For example, the scope of the universal quantifier in the statement

$$\forall x\text{: (F(x)} \cap \text{P(x))}$$

is (F(x) $\cap$ P(x)), likewise the scope of the existential quantifier in the statement

$$\exists x\text{: F(x)} \longrightarrow \text{P(y)}$$

is F(x).

Examples: Find the scopes of quantifiers in the following statements

1. $\forall y$: ~ P(y)
2. $\exists x$, $\forall x$: P(x) $\longleftrightarrow$ F(y)
3. $\forall x$: $F(x) \cap P(y)$
4.

5.

2.11 Bound and Free Variables

In a quantified statement, a variable can play 2 different roles i.e. either a it is a bound variable or a free variable.

2.11.1 Bound Variable

A bound variable can simply be defined as a variable that is quantified in a statement, for example in the statement $\forall x$: P(x) $\longrightarrow$ F(y), x is a bound variable since it is quantified by $\forall$

2.11.2 Free Variable

A free variable can simply be defined as a variable that is not quantified in a statement, for example in the statement $\forall x$: P(x) $\longrightarrow$ F(y), y is a free variable since it is not quantified.

Examples: Identify the bound and free variables in the following statements

1. $\forall x$: $(x \cup y) \cap (\exists y : \tilde{}y \cup x)$
   Solution:
   The first x is bounded by $\forall x$
   The first y is free
   The second x is bounded by $\forall x$
   The second y is bounded by $\exists y$
2. $\forall x$: $P(x, y) \cap (z \cup y \longrightarrow x)$
3. $\exists x$: $(x \cap \tilde{}y) \longleftrightarrow (y \longrightarrow z)$
4. $\forall y, existsx$: $P(x) \cap P(y) \cup P(z)$
5. $\forall x, \exists y$: $(x = y + y) \cup (x = y + y + 1)$

2.12 Formula

A formula in logic simply means a statement which consists of variables and connectives and also has a truth value. A formula in logic may also be called a propositional expression, a sentence, or a sentential formula. Example of a formula is $(\mathbf{p} \cap \tilde{}\mathbf{q}) \longrightarrow (\mathbf{p} \cup \mathbf{q})$,

2.12.1 Argument

An argument is a list of statements called the premises followed by a conclusion. For example,

p := Every student in ND 1 does Maths
q := Umar is in ND 1 another example is

Therefore, **r** := Umar does Maths

p := Federal Polytechnic Nasarawa is in Nasarawa
q := Nasarawa is in Nigeria
r := Nigeria is in West Africa
s := West Africa is in Africa

Therefore, **t** := Federal Polytechnic Nasarawa is in Africa
Generally, an argument can be written in the form

$$\underbrace{p_1 \cap p_2 \cap p_3 \cap ... \cap p_n}_{premises} \longrightarrow \underbrace{q}_{conclusion}$$

2.12.2 Validity of an Argument

An argument

$$(p_1 \cap p_2 \cap p_3 \cap ... \cap p_n \longrightarrow \mathbf{q}$$

) is said to be *valid* if the statement is a tautology. In other words, if the truth table of the statement gives **T** in the last column.

Examples: Identify the premises and conclusion and check whether the following arguments are valid or not:

1. If a man is a bachelor, he is unhappy. If a man is unhappy, he is depressed. Bachelor is depressed.
2. If I play, I cant study. Either I play or I study maths. I studied Maths.

3. All Fulanis are slim. Maryam is a Fulani. Maryam is slim.
4. He is in my class. He is a talkative. Every one in my class is a talkative.
5. All girls like paintings. None of my friends like painting. All my friends are boys. **Solution:**

Chapter 3

Permutation and Combination

3.1 Permutation

Permutation of objects means all possible arrangement of of objects where the order is important. Take for example three letters **A**, **B** and **C**. These letters can be arranged as follows:

A B C
A C B
B A C
B C A
C A B
C B A

The above arrangement was done while putting emphasis on the order of arrangement, this is why **A B C** is one arrangement and **A C B** is another different one. This is called permutation. Therefore, there are 6 different ways of permuting (arranging) three letters.
Now consider the permutation of four letters **A B C** and **D**. This can be done in the following ways

A B C D	**B A C D**	**C A B D**	**D A B C**
A B D C	**B A D C**	**C A D B**	**D A C B**
A C B D	**B C A D**	**C B A D**	**D B A C**
A C D B	**B C D A**	**C B D A**	**D B C A**
A D B C	**B D A C**	**C D A B**	**D C A B**
A D C B	**B D C A**	**C D B A**	**D C B A**

If we continue arrangement in the above manner, we observe the following

S/N	No. of Objects	No. of Ways of Arrangements
1	3	$6 = 3 \times 2 \times 1 = 3!$
2	4	$24 = 4 \times 3 \times 2 \times 1 = 4!$
3	5	$120 = 5 \times 4 \times 3 \times 2 \times 1 = 5!$
4	6	$720 = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$

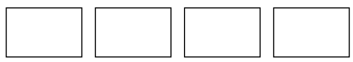
Therefore, the number of ways of arranging n objects is $n!$.

3.1.1 Permutation Formula

The fundamental theorem of counting states that:

> "if a task can be performed in n_1 ways, a second task in n_2 ways and a third task in n_3 ways and so on, then the total number of distinct ways of performing all tasks together is $n_1 \times n_2 \times n_3 \times ...$"

Consider 5 different books say A, B, C, D and E. Suppose we want to arrange 4 out of the five books. To do this, we assume there are 4 spaces for the arrangement, i.e.



We start by putting one of the books in the first box (space), there are 5 different ways doing this since you have 5 books to select from. After choosing a book we are now left with 4 books to choose from. We then move to the second box.
To fill the second box, there are 4 ways of doing it because we have 4 books left to choose from. After choosing a book, we are left with only 3 books. We then move to the third box.
To fill the third box, there are 3 ways of doing it because we have 3 books left to choose from. After choosing a book, we are left with only 2 books. We then move to the fourth box.
Finally, to fill the fourth box, there are 2 ways of doing it because we have 2 books left to choose from. After choosing a book, we now use the fundamental theorem of counting to calculate the number of ways

of arranging 4 books out of the five books, Thus we have,
5 ways of arranging the first book, 4 ways of arranging the second book, 3 ways of arranging the third book and finally 2 ways of arranging the fourth book. Therefore the number of ways of arranging 4 books out of 5 books is given by

$$5 \times 4 \times 3 \times 2 = 120$$

We write ${}^5P_4$ to represent the number of ways of arranging 4 out of 5 objects, hence

$${}^5P_4 = 5 \times 4 \times 3 \times 2 = 120$$

${}^5P_4$ is pronounced "5 permutation 4"
In a similar manner, we have the following,
The number of ways of arranging 4 out of 5 objects is

$${}^5P_4 = 5 \times 4 \times 3 \times 2 = 120$$

The number of ways of arranging 3 out of 5 objects is

$${}^5P_3 = 5 \times 4 \times 3 = 60$$

The number of ways of arranging 2 out of 5 objects is

$${}^5P_2 = 5 \times 4 = 20$$

The number of ways of arranging 5 out of 7 objects is

$${}^7P_5 = 7 \times 6 \times 5 \times 4 \times 3 = 2520$$

and so on.
Now, lets generalize, consider

$${}^7P_3$$

we have

$$\begin{aligned} {}^7P_3 &= 7 \times 6 \times 5 \\ &= 7 \times 6 \times 5 \times \left(\tfrac{4\times3\times2\times1}{4\times3\times2\times1}\right) \\ &= \tfrac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1} \\ &= \tfrac{7!}{4!} \\ &= \tfrac{7!}{(7-3)!} \end{aligned}$$

In a similar manner

$$^5P_4 = \frac{5!}{(5-4)!}$$
$$^5P_3 = \frac{5!}{(5-3)!}$$
$$^7P_5 = \frac{7!}{(7-5)!}$$

and so on.

This takes us to the following Permutation theorem.

The theorem states that:

> "The number of ways of arranging r objects out of n objects written as ^nP_r is given by $^nP_r = \frac{n!}{(n-r)!}$"

Proof:

Recall that $^nP_r = n(n-1)(n-2)...(n-(r-1))$

Thus we have

$$\begin{aligned} ^nP_r &= n(n-1)(n-2)...(n-(r-1)) \\ &= n(n-1)(n-2)...(n-(r-1)) \times \frac{(n-r)(n-r-1)(n-r-2)...\times 3\times 2\times 1}{(n-r)(n-r-1)(n-r-2)...\times 3\times 2\times 1} \\ &= \frac{n(n-1)(n-2)...(n-(r-1))(n-r)(n-r-1)(n-r-2)...\times 3\times 2\times 1}{(n-r)(n-r-1)(n-r-2)...\times 3\times 2\times 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Hence proved.

Examples:

1. Evaluate the following:
   a. $^{10}P_6$
   b. $^8P_4$
   c. $^5P_5$
2. Show that $^nP_r = (n-r+1) \times ^n P_{r-1}$
3. Box 1 contains the letters A, B, C, D, E, F, G whereas Box 2 contains the letters W, X, Y, Z. How many letter codes can be constructed using
   a. 3 letters from box 1 and 2 letters from box 2?
   b. 2 letters from box 1 and 3 letters from box 2?
   c. Why are the above not equal?

4. How many three - letter initials can possibly be formed using only the letters M, N, Q, R, S, and T, if
   a. Repetitions are allowed?
   b. Repetitions are not allowed?
5. In how many ways can 7 people be put in ten seats?

3.1.2 Permutation of Repeated Objects

In the permutation of n objects and within the n objects there are some that are repeated, for example in the word

NECESSARY

The letter **E** is repeated twice so also is the letter **S**, if there are n_1, n_2, n_3, . . . ,n_k repetitions, then the number of ways of arranging the the n objects where there are n_1, n_2, n_3, . . . ,n_k repetitions is given by

$$\frac{n!}{n_1!n_2!n_3!...n_k!}$$

Examples:

1. How many different arrangements can be made using the letters of the following words
   a. MATHEMATICS
   b. HIPPOPOTEMUS
   c. MATHEMATICIANS
   d. MISSISSIPPI
2. A team consists of 5 ND 1 student, 3 ND 2 students and 4 HND 1 students. How many ways can they be arranged on a line if the students of the same class must be together?
3. How many different car numbers can be formed using the 10 letters (A to J) and 8 numbers (0 to 7) if each number must have 3 letters and 3 digits with the letters at the beginning.

4. In how many ways can 3 prizes be won by 12 students of a class
   a. If no student can win more than on prize
   b. If no restriction is placed on the winning of a prize
   c. If only a student wins exactly 3 prizes

3.1.3 Permutation with Restrictions

Supposing we are to arrange n objects in a row such that r objects out of the n objects are to be together is given by the formula

$$[n-(r-1)]!r!$$

and if r objects out of the n objects are not to be together is given by the formula

$$[n-(r-1)](n-r)(n-r)!$$

Examples:

1. **Solution:**

2. **Solution:**

NOTE THAT: $0! = 1! = 1$.

3.1.4 Circular Permutation

Supposing n objects are to be arranged in a ring or circle, then this can be done in $(n-1)!$ ways and if the objects are fixed such that the ring can be turned over, then the required permutations becomes $\frac{(n-1)!}{2}$

The arrangement of n distinct objects in a circle such that r out of the objects are together is given by

$$(n-r)!r!$$

and if the r objects are not together is given by

$$(n-1)[n-(r+1)][n-(r+1)]!$$

Examples:

1. In how many ways can 7 people be seated in a round table?
2. In how many ways can a person arrange 5 bids in a circle?
3. In how many ways can 5 people be arranged in a circle such that two people must sit together?
4. In how many ways can 8 people sit together on a round table so that
   a. 2 people must sit together?
   b. 2 people must not sit together?

3.2 Combination

Combination of objects means all possible selection of of objects. In combination, the order is not important. Take for example four letters **A**, **B**, **C** and **D** and suppose we want to select 3 out of the four letters, then the selection can be made as follows:
The selections

A B C, **A C B**, **B A C**, **B C A**, **C A B** and **C B A**

are all counted as only 1 selection because the order is not important. So therefore, for the selection of 3 out of the 4 letters we have only 4 selections, which are

A B C
A B D
A C D
B C D

In a similar manner, for the selection of 2 out of the 4 letters we have only 6 selections, which are

A B
A C
A D
B C
B D
C D

Also, for the selection of 4 out of 5 letters **A, B, C, D** and **E**, we have only 5 selections, which are

A B C D
A B C E
A B D E
A C D E
B C D E

If we observe from the above, we can deduce the following
Supposing we let ${}^4C_2$ to represent the number of ways of selecting 2 out of 4 objects (pronounced as 4 combination 2), then we have
${}^4C_2 = 6$,
${}^4C_3 = 4$ and
${}^5C_4 = 5$.
Lets consider the first i. e.

$${}^4C_2 = 6$$

We know that

$${}^4P_2 = \frac{4!}{(4-2)!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 12$$

combining the above equations we get

$${}^4C_2 = 6 = \frac{12}{2} = \frac{{}^4P_2}{2!} = {}^4P_2 \times \frac{1}{2!} = \frac{4!}{(4-2)!2!}$$

Similarly,

$${}^4C_3 = 4 = \frac{24}{6} = \frac{{}^4P_3}{3!} = {}^4P_3 \times \frac{1}{3!} = \frac{4!}{(4-3)!3!}$$

Also,

$${}^5C_4 = 5 = \frac{120}{24} = \frac{{}^5P_4}{4!} = {}^5P_4 \times \frac{1}{4!} = \frac{5!}{(5-4)!4!}$$

Generally, the number of ways of selecting r objects out of n objects is given by the formula

$$^nC_r = \frac{n!}{(n-r)!r!}$$

Examples:

1. Evaluate the following
   a. $^6C_3$
   b. $^5C_3$
   c. $^8C_2$
   d. $^7C_0$
   e. $^{10}C_{10}$
2. A committee of 6 members (4 males and 2 females) is to be formed from 10 males and 4 females. In how many ways can this be done?
3. A committee of 5 members is to be formed from 3 females and 4 males. In how many ways can it be formed if
   a. at least one female is included as a member?
   b. at least one ,male is included as a member?
4. Prove the following
   a. $^nC_{n-r} =^n C_r$
   b. $^nC_r +^n C_{r+1} =^{n+1} C_{r+1}$
5. In a class of 14 boys and 10 girls, a committee of 7 is to be formed. How many committees are possible if
   a. anybody can serve in the committee?
   b. the committee is to have exactly 4 boys?
   c. the committee is to contain at least 4 boys
6. Simplify the following
   a. $^{10}C_7 +^{10} C_6$
   a. $^7C_{r+1} +^7 C_r$
   a. $^{2r}C_r +^{2r} C_{r-1}$

Chapter 4

Binomial Theorem of Algebraic Expressions

4.1 Mathematical Induction

Mathematical induction is a special way of proving things like statements, laws, theorems, formula etc. It is mostly used to establish a given statement/formula for all natural numbers. The idea behind Mathematical induction is simple, it follows three steps as follows:

STEP 1: Prove that the statement/formula is true for the first number

STEP 2: Assume it is true for any number k

STEP 3: Then show that it is true for the next number after k i.e. $k + 1$

If the above steps are proved, then that simply means that it is true for all natural numbers.

> **Examples:** Prove the following using Mathematical Induction
>
> 1. $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$
> 2. $3^n - 1$ is even
> 3. $5 + 10 + 15 + ... + 5n = \frac{5n(n+1)}{2}$
> 4. $3^n - 2n - 1$ is divisible by 4
> 5. $1 + 3 + 5 + .. + (2n - 1) = n^2$
> 6. $4^{2n} - 1$ is divisible by 3

7. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+...+\frac{n}{2^n}=2-\frac{n+2}{2^n}$

8. $1^2+2^2+3^2+...+n^2=\frac{1}{6}n(n+1)(2n+1)$

4.2 The Pascal Triangle

The Pascal triangle Method is an easy way for solving a binomial expansion with a small index. A binomial is an algebraic expression of the sum or the difference of two terms for example $(a+b),(2+x),(y+x)$, etc.

When a binomial has an index (power), there is a need to expand it to get the actual expression for example, consider the binomial

$$(a+b)^2$$

This can be expanded as follows:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+ab+ba+b^2\\ &= a^2+ab+ab+b^2 \text{ since } ba=ab\\ &= a^2+2ab+b^2\end{aligned}$$

In a similar expansion for $(a+b)^3$we have

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)(a+b)\\ &= (a^2+2ab+b^2)(a+b)\\ &= a^3+a^2b+2a^2b+2ab^2+b^2a+b^3\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

Expanding in this way upto the index 5, we obtain the following table

$$\begin{aligned}(a+b)^1 &= a+b\\ (a+b)^2 &= a^2+2ab+b^2\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\ (a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\ (a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\end{aligned}$$

From the table above we can observe the following

1. The powers of a is decreasing by 1 across the terms
2. The powers of b is increasing by 1 across the terms
3. There is a special pattern made by the coefficients of the terms in the expansion as follows:

Binomial		Coefficients
$(a+b)^1$	=	1 1
$(a+b)^2$	=	1 2 1
$(a+b)^3$	=	1 3 3 1
$(a+b)^4$	=	1 4 6 4 1
$(a+b)^5$	=	1 5 10 10 5 1

The coefficient in the above table forms a Triangular pattern known as the Pascal Triangle. In the Pascal triangle, coefficients of a particular power can be found using the coefficient of the previous power. The following table illustrates how the Pascal triangle is generates

$$\begin{array}{ccccccccccc}
 & & & 1 & & & & 1 & & & \\
 & & & & \searrow & \text{add} & \swarrow & & & & \\
 & & 1 & & & 2 & & & & 1 & \\
 & & & \searrow & \text{add} & \swarrow & \searrow & \text{add} & \swarrow & & \\
1 & & & 3 & & & & 3 & & & 1 \\
\vdots & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots \\
\vdots & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots \\
\vdots & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Supposing you are asked to expand the binomial $(x+2)^5$ using the Pascal Triangle methods, the following steps should be followed,

Step 1: Bring out the coefficients for the power 5 using the Pascal triangle and these are the following coefficients

$$1 \qquad 5 \qquad 10 \qquad 10 \qquad 5 \qquad 1$$

Step 2: Arrange the coefficients in the following format

$$1 \qquad +5 \qquad +10 \qquad +10 \qquad +5 \qquad +1$$

Step 3: Pick the first term in the binomial and multiply each coefficient with it starting from the first coefficient with the power and as you move across the terms in the expansion reduce one from the power as follows (note that the first term in the binomial expression is x)

$$1 \times x^5 \qquad +5 \times x^4 \qquad +10 \times x^3 \qquad +10 \times x^2 \qquad +5 \times x^1 \qquad +1 \times x^0$$

Step 4: Now pick the second term in the binomial and multiply it across the terms but in this case starting with power zero and adding 1 to the power when multiplying across the terms of the expansion as follows (note that the second term the binomial expression is +2)

$$1\times x^5\times 2^0+5\times x^4\times 2^1+10\times x^3\times 2^2+10\times x^2\times 2^3+5\times x^1\times 2^4+1\times x^0\times 2^5$$

Step 5: Simplify the expression, thus we have

$$\begin{aligned}(x+2)^5 &= x^5\times 1+5x^4\times 2+10x^3\times 4+10x^2\times 8+5x\times 16+1\times 32\\ &= x^5+10x^4+40x^3+80x^2+80x+32\end{aligned}$$

NOTE: The following are worthy to note when expanding a binomial expression using the Pascal Triangle

- When a term in a binomial has a negative sign for example $(x-2)^3, (y-3)^4, (4-x)^7$ etc, then when picking the terms for the expansion, the negative sign must be used i.e. in the binomial $(x-3)^4$, the first term is x while the second term is -3 not 3.
- When you are asked to expand a binomial so that one of the terms should be in ascending powers or descending powers, then you will have to rearrange if the need arise. For example in the expansion of $(a+b)^4$ we obtain

  $$(a+b)^4 = (1)a^4(b^0)+(4)a^3(b^1)+(6)a^2(b^2)+(4)a^1(b^3)+(1)a^0(b^4)$$

  we observe that the power of a is decreasing while the power of b is increasing, therefore the expansion is in descending powers of a and also ascending powers of b.
  Another way we can express the binomial $(a+b)^4$ is $(b+a)^4$ since

  $$(a+b)^4 = (b+a)^4$$

  Now expanding $(b+a)^4$ we get

  $$(b+a)^4 = (1)b^4(a^0)+(4)b^3(a^1)+(6)b^2(a^2)+(4)b^1(a^3)+(1)b^0(a^4)$$

  the expansion is now in descending powers of b and ascending powers of a. So the position of the term is responsible for either ascending or descending power of the term in the expansion. You can always interchange the positions of the terms to suit the question. Some of these are done below

$$(a-b)^5 = (-b+a)^5$$
$$(2+x)^4 = (x+2)^6$$
$$(7-y)^8 = (-y+7)^8$$
$$(-3-b)^5 = (-b-3)^5$$

etc.

- The Pascal triangle method is mostly used if the power of the binomial is less than 10. If the power exceeds 10, then it becomes difficult for someone to use this method because bringing out the coefficients may be very difficult for example if you are given $(a+5)^{33}$ to expand. Bringing out the triangle of to the power 33 may pose alot of problems. A more accommodating method for binomial expansion with a large index is the Binomial Theorem.

Examples: Use the Pascal Triangle method to expand the following binomials

1. $(3-2x)^5$
2. $(\frac{x}{2}-5)^7$
3. $(x+3)^6$ in ascending powers of x
4. $(5-3x)^9$ in descending powers of x
5. $(2-\frac{1}{x})^5$

4.3 Binomial Theorem for Positive Integer Indices

Binomial theorem for positive integer indices is a simplified way of expanding binomial express especially the ones with large indices like $(a+b)^{59}$ etc. The theorem was formed using a special way of obtaining

the coefficients. This special way is explained below.
Let us consider the coefficients of the binomial expression with powers 3, 4 and 5 i.e.

Binomial												
$(a+b)^3$	=				1		3		3		1	
$(a+b)^4$	=			1		4		6		4		1
$(a+b)^5$	=	1		5		10		10		5		1

(Coefficients)

Now observe the following

Binomial		Coefficients					
$(a+b)^3$	:	$1 = {}^3C_0$	$3 = {}^3C_1$	$3 = {}^3C_2$	$1 = {}^3C_3$		
$(a+b)^4$	:	$1 = {}^4C_0$	$4 = {}^4C_1$	$6 = {}^4C_2$	$4 = {}^4C_3$	$1 = {}^4C_4$	
$(a+b)^5$	:	$1 = {}^5C_0$	$5 = {}^5C_1$	$10 = {}^5C_2$	$10 = {}^5C_3$	$5 = {}^5C_4$	$1 = {}^5C_5$

Therefore using this idea, the coefficients for any power can be obtained without the Pascal Triangle. For example, the coefficients of the power 6 will be

$${}^6C_0,\ {}^6C_1,\ {}^6C_2,\ {}^6C_3,\ {}^6C_4,\ {}^6C_5,\ \text{and}\ {}^6C_6$$

Generalizing we get the Binomial theorem for positive integer indices as follows

Theorem: (Binomial Theorem for Positive Integer Indices): The theorem states that

$$(a+b)^n = \sum_{r=0}^{r=n} {}^nC_r a^{n-r} b^r$$

where n is a positive integer.

Proof: We are going to use Mathematical Induction to prove this theorem

Step 1:

Test for $n = 1$ to see whether

$$(a+b)^n = \sum_{r=0}^{r=n} {}^nC_r a^{n-r} b^r$$

thus we have the left hand side (LHS) of the equation to be

$$(a+b)^1 = a + b$$

and the right hand side (RHS) is

$$\begin{aligned}\sum_{r=0}^{r=1} {}^1C_r a^{1-r} b^r &= {}^1C_0 a^{1-0} b^0 +^1 C_1 a^{1-1} b^1 \\ &= \text{1(a)(1)+1(1)(b)} \\ &= \text{a+b}\end{aligned}$$

Therefore the LHS = $(a+b)^n = \sum_{r=0}^{r=n} {}^nC_r a^{n-r} b^r$ = RHS, hence it is true for $n = 1$

Step 2:

Assuming it is true for $n = k$ where k is an integer, then

$$(a+b)^k = \sum_{r=0}^{r=k} {}^kC_r a^{k-r} b^r$$

Step 3: For $n = k + 1$ we have

$$\begin{aligned}(a+b)^{k+1} &= (a+b)^k \cdot (a+b) \\ &= \sum_{r=0}^{r=k} {}^kC_r a^{k-r} b^r \cdot (a+b) \\ &= (a+b) \cdot \sum_{r=0}^{r=k} {}^kC_r a^{k-r} b^r\end{aligned}$$

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Therefore, supposing we want to expand $(a+b)^4$ we first of all bring out the coefficient just as we did in the Pascal triangle method but in this case, we are going to use the binomial coefficient method as follows; the coefficients for the expansion of $(a+b)^4$ are

$$^4C_0, \quad ^4C_1, \quad ^4C_2, \quad ^4C_3, \quad ^4C_4$$

the remaining process is the same as that of Pascal triangle method, thus we have the following

$$(a+b)^4 = {}^4C_0 a^4 b^0 + {}^4C_1 a^3 b^1 + {}^4C_2 a^2 b^2 + {}^4C_3 a^1 b^3 + {}^4C_4 a^0 b^4$$

but we know that

$$^4C_0 = 1, \quad ^4C_1 = 4, \quad ^4C_2 = 6, \quad ^4C_3 = 4, \quad ^4C_4 = 1$$

thus

$$(a+b)^4 = a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + b^4$$

Similarly for the expansion of $(a+b)^5$, we have the following coefficients

$$^5C_0, \quad ^5C_1, \quad ^5C_2, \quad ^5C_3, \quad ^5C_4, \quad ^5C_5$$

therefore

$(a+b)^5 = {}^5C_0a^5b^0 + {}^5C_1a^4b^1 + {}^5C_2a^3b^2 + {}^5C_3a^2b^3 + {}^5C_4a^1b^4 + {}^5C_5a^0b^5$

but

$${}^5C_0 = 1, \quad {}^5C_1 = 5, \quad {}^5C_2 = 10, \quad {}^5C_3 = 10, \quad {}^5C_4 = 5, \quad {}^5C_5 = 1$$

thus

$$(a+b)^5 = a^5 + 5a^4b^1 + 10a^3b^2 + 103a^2b^3 + 5a^1b^4 + b^5$$

Properties of Binomial Expansion

The following are some properties of binomial expansion $(a+b)^n$ where n is a positive integer

1. The number of terms in the expansion is (n+1) which is one more than the index.
2. Every term in the expansion of $(a+b)^n$ can be written in a compact form as
$$ {}^nC_ra^{n-r}b^r $$
depending on the value of r.
If $r = 0$, we obtain the first term i.e. ${}^nC_0a^{n-0}b^0 = 1 \cdot a^n \cdot 1 = a^n$
If $r = 1$, we obtain the second term i.e. ${}^nC_1a^{n-1}b^1 = n \cdot a^{n-1} \cdot b = na^{n-1}b$
and so on.
3. The first and the last terms are a^n and b^n respectively.
4. The sum of the coefficients is
$$ {}^nC_0 + {}^nC_1{}^nC_2 + ... + {}^nC_n = 2^n $$
5. ${}^nC_0 = {}^nC_n$ and ${}^nC_1 = n$
6. Progressing from the first term to the last, the exponent of a decreases by 1 from term to term while the exponent of b increases by 1. In addition, the sum of the exponents of a and b in each term is n.

Examples: Use the Binomial theorem to find the first 6 terms of the following binomials

1. $(x^2 - \frac{1}{x})^2 0$
2. $(\frac{x}{2} - \frac{y}{3})^1 5$ in ascending powers of x
3. $(2 - \frac{x}{2})^1 2$
4. $(a - 5)(a + 3)^1 0$
5. $(x + 2y^2)^1 2$

4.3.1 Finding the Constant term (independent of x) in an Expansion

Consider the following expansion

$$(x + 2)^2 = x^2 + 4x + 4$$

The term 4 in the expansion is called the constant term because there is no x attached to it. It is possible to obtain the constant term in an expansion without expanding the binomial. This is done using the compact form of the binomial $(a + b)^n$ which is

$$^nC_r a^{n-r} b^r$$

. To get the constant term, we look for the value of r (in the compact form of a binomial expression) that will make the power of x be 0. We then substitute it and evaluate, for example, to find the constant term in the expansion of

$$(x + 6)^5$$

, we first of all find the compact form which is

$$^nC_r a^{n-r} b^r$$

but comparing $(x + 6)^5$ with $(a + b)^n$, we get $a = x$, $b = 6$ and $n = 5$, thus we have the compact form as

$$^nC_r a^{n-r} b^r = {}^5C_r x^{5-r} 6^r$$

We can see that the power of x in the compact form is $5 - r$ and the value of r that will make this power 0 is $r = 5$, we then substitute it to obtain the constant term as

$$^{5}C_{5}x^{5-5}6^{5} = 1 \cdot x^{0} \cdot 6^{5} = 6^{5}$$

Another example is finding the constant term of the expansion of

$$(x - \frac{2}{x})^{4}$$

In this case we first of all obtain the compact form, we have $a = x$, $b = \frac{-2}{x}$ and $n = 4$, thus the compact form is

$$^{n}C_{r}a^{n-r}b^{r} = {}^{4}C_{r}x^{4-r}(\frac{-2}{x})^{r}$$

If you observe, we cannot get the power of x since there are 2 xs in the compact form, so what we will do is to use the law of indices to try to bring the xs together before taking the power, thus we have

$$\begin{aligned} ^{n}C_{r}a^{n-r}b^{r} &= {}^{4}C_{r}x^{4-r}(\tfrac{-2}{x})^{r} \\ &= {}^{4}C_{r}x^{4-r}\tfrac{(-2)^{r}}{(x)^{r}} \\ &= {}^{4}C_{r}\tfrac{x^{4-r}}{(x)^{r}}(-2)^{r} \\ &= {}^{4}C_{r}x^{4-r-r}(-2)^{r} \\ &= {}^{4}C_{r}x^{4-2r}(-2)^{r} \end{aligned}$$

Now we have the power of x as $4 - 2r$, so to get the value of r that will make this power 0, we equate the power to 0 and find the value of r, thus we have

$$\begin{aligned} 4 - 2r &= 0 \\ \Rightarrow \quad -2r &= -4 \\ \Rightarrow \quad r &= \tfrac{-4}{-2} \\ \Rightarrow \quad r &= 2 \end{aligned}$$

we then substitute the value of $r = -2$ to obtain the constant term of the expansion, thus

$$\begin{aligned} ^{4}C_{r}x^{4-r}(\tfrac{-2}{x})^{r} &= {}^{4}C_{r}x^{4-2r}(-2)^{r} \\ &= {}^{4}C_{2}x^{4-2(2)}(-2)^{2}, \text{ substituting } r = 2 \\ &= 6 \cdot x^{0} \cdot 4 \\ &= 6 \cdot 4 \\ &= 24 \end{aligned}$$

Thus the constant term is 24

Examples: Find the constant term (term independent of x) in the expansion of each of the following binomial

1. $(x^2 + \frac{3}{x})^9$
2. $(\frac{1}{x} - 2x)^12$
3. $(5x - \frac{x^2}{2})^6$
4. $(2x - \frac{8}{x^2})^5$
5. $(\frac{1}{x} - \frac{2}{x})^9$

4.3.2 Finding the Coefficient of a Particular term in an Expansion

Finding the coefficient of a term in an expansion is similar to finding the constant term, the difference is that instead of finding r that will making the power of x to be 0, we find the value of r that will make the power of x to be the power of x in the required term. For example, consider the expansion of $(x + 2)^4$ is

$$(x + 2)^4 = x^4 + 8x^3 + 24x^2 + 24x + 16$$

The constant term is 16
The coefficient of x in the expansion is 24
The coefficient of x^2 in the expansion is 24
The coefficient of x^3 in the expansion is 8 and
The coefficient of x^4 in the expansion is 1
The coefficients can be obtained using the compact form, for instance if we want to get the coefficient of x^3 in the expansion of $(x + 2)^4$, we first of all find the compact form then find the value of r that will make the power of x to be 3. We then substitute it and find the coefficient. Hence we have the compact form as

$${}^4C_r x^{4-r}(2)^r$$

since we want the coefficient of x^3, we will look for the value of r that will make the power of x in the compact form (i.e. $4 - r$) be equal to 3. Clearly, $r = 1$ will make $4 - r$ to be 3, thus we substitute

$$\begin{aligned} {}^4C_r x^{4-r}(2)^r &= {}^4C_1 x^{4-1}(2)^1, \text{ substituting } r = 1 \\ &= 4 \cdot x^3 \cdot 2 \\ &= 8x^3 \end{aligned}$$

Therefore the coefficient of x^3 in the expansion of $(x+2)^4$ is 8.

Examples:

1. Find the coefficient of x^2 and x^6 in the expansion of $(\frac{1}{x} - 2x)^{10}$
2. Calculate the coefficient of x^4 in the expansion of $(x^2 - \frac{2}{x})^5$
3. Determine the coefficient of x^{-2} in the expansion of $(x - \frac{1}{x})^4$
4. Find the term in x^9y^6 in the expansion of $(x + 2y^2)^12$
5. Find the coefficient of x^2 in the expansion of $(\frac{x^2}{4} - 2x)^9$

4.4 Binomial Theorem for Negative and Fractional Indices

Supposing we are asked to use the binomial theorem to expand expressions such as

$$\frac{1}{(x+2)^4} = (x+2)^{-4}$$
$$\sqrt{(2+x)} = (2+x)^{\frac{1}{2}}$$
$$\sqrt[3]{(x-1)} = (x-1)^{\frac{1}{3}}$$

then, this is not going to be possible using the binomial theorem for positive integer indices because, if we take $(x+2)^{-4}$ for example, we will have to bring out the coefficients which are

$${}^{-4}C_0, {}^{-4}C_1, {}^{-4}C_2, ...$$

but in evaluating ${}^{-4}C_0$, we will have to find $(-4)!$ which does not exist since we can only find the factorial of a positive number.

This poses a problem, that is, how to expand a binomial with a negation or fractional indices. To solve this problem, the binomial theorem for positive integer indices was modified to accomodate the negative and fraction indices. This modification is illustrated as follows.

The binomial theorem for positive integer indices is

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + {}^nC_3a^{n-3}b^3 + ... + {}^nC_nb^n$$

The main constraint that deos not allow us to use this theorem for negative or fractional index is the fact that we can not find the factorial of a negative number or a fraction therefore the combination cannot be found. What we will do is to find a way to find ${}^nC_r$ without using factorial and this is found below

${}^nC_0 = 1$ this is a known fact that any number combination 0 is 1 (can be proven)

${}^nC_1 = n$ this is also a known fact that any number combination 1 is that number (can be proven)

${}^nC_2 = \frac{n!}{(n-2)! \cdot 2!} = \frac{n(n-1)(n-2)!}{(n-2)! \cdot 2!} = \frac{n(n-1)}{2!}$

${}^nC_3 = \frac{n!}{(n-3)! \cdot 3!} = \frac{n(n-1)(n-2)(n-3)!}{(n-3)! \cdot 3!} = \frac{n(n-1)(n-2)}{3!}$

${}^nC_4 = \frac{n!}{(n-4)! \cdot 4!} = \frac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)! \cdot 4!} = \frac{n(n-1)(n-2)(n-4)}{4!}$

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${}^nC_n = 1$ this is a known fact that any number combination itself is 1 (can be proven)

Replacing the above in the binomial theorem for positive integer indices we get the binomial theorem for negative or fractional indices as follows

$$(a+b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + ... + b^n$$

1. Write down the first three terms of the expansion of $\frac{1}{(1=X)^2}$
2. Find the first 4 terms of the power series expansion of $(1 - \frac{x}{2})^{-\frac{1}{2}}$
3. Expand $(8 + 3x)^{\frac{1}{3}}$ in ascending powers of x as far as the term in x^2.
4. Obtain the first 4 terms of the expansion of $\sqrt{x^2 - 5}$
5. Find the first 5 terms of the expansion of $\frac{1}{(x-3)^5}$

4.5 Application of Binomial Theorem to Approximation

Expression like $(1+x)^n$ can be approximated by using the first few terms for the expansion if x is sufficiently small because higher powers of x can be negligible. Therefore if x is sufficiently small, the expansion of $(a+x)^n$ canhave the following approximations

$$\text{Linear approximation } (a+x)^n \approx a^n + na^{n-1}x$$
$$\text{Quadratic approximation } (a+x)^n \approx a^n + na^{n-1}x + \tfrac{n(n-1)}{2!}a^{n-2}x^2$$

Examples:

1. For example, write down the first 4 terms of the expansion of $(1-x)^5$ in ascending powers of x. Use your answer to find $(0.998)^5$ correct to 3 decimal places.

   Solution:

   Expanding $(1-x)^5$ we have
   $$\begin{aligned}(1-x)^5 &\approx {}^5C_0 1^5 + {}^5C_1 1^4(-x) + {}^5C_2 1^3(-x)^2 + {}^5C_3 1^2(-x)^3 \\ &= 1 - 5x + 10x^2 - 10x^3\end{aligned}$$
   To approximate $(0.998)^5$ using the expansion, compare it with $(1-x)^5$ and find the value of x, thus we have
   $$\begin{aligned}(0.998)^5 &= (1-x)^5 \\ \Rightarrow \quad 0.998 &= 1-x \\ \Rightarrow \quad x &= 1-0.998 \\ \Rightarrow \quad x &= 0.002\end{aligned}$$
   therefore we have $x = 0.002$, hence

   $(1-x)^5 \approx 1-5x+10x^2-10x^3$ substituting the value of x we get
   $$\begin{aligned}(0.998)^5 &= (1-0.002)^5 \approx 1 - 5(0.002) + 10(0.002)^2 - 10(0.002)^3 \\ &= 1 - 0.01 + 0.00004 - 0.00000008 \\ &= 0.99003992 \\ &= 0.990 \text{ to 3 d.p.}\end{aligned}$$

2. Find the first four terms of the expansion of $(1+\frac{x}{2})^{10}$ in ascending powers of x. Hence, find the value of (1.002)610 correct to 5 d.p.

3. Using the first 3 terms of the binomial series for $(1+x)^{\frac{1}{3}}$, find $\sqrt[3]{0.99}$ correct to 3 d.p.

4. Find the first 3 terms of the expansion of $(1-x)^{-\frac{1}{2}}$ and use it

to calculate $\frac{1}{\sqrt{1.01}}$ correct to 3 d.p.

5. Without using tables or calculator, find the value of $(2+\sqrt{5})^4 + (2-\sqrt{5})$

Chapter 5

Matrices and Determinants

5.1 Matrices

A matrix is an ordered array of numbers arranged in rows and columns for example

$$\begin{pmatrix} 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 7 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 & 8 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 5 & 7 \\ 4 & 5 & 6 & 0 & 1 \\ 7 & 8 & 9 & 2 & 1 \\ 0 & 5 & 1 & 8 & 6 \end{pmatrix}$$

etc.

Each entry in the matrix is called an element. A general way of representing a matrix with n column and m rows is

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

The subscript shows the position of the entry for example a_{45} mean the entry in 4th row and 5th column.

5.1.1 Order of a Matrix

The order (or dimension) of a matrix is its number of rows by its number of columns. By convention, rows are listed first; and columns, second. Thus, we would say that the order (or dimension) of the matrix with 3 rows and 4 columns is 3 x 4, pronounced 3 by 4. The table below illustrates the order of matrices

S/N	No.of rows	No. of columns	Matrix	Dimension
1	$\begin{pmatrix} 7 & 8 & 9 \end{pmatrix}$	1	3	1×3
2	$\begin{pmatrix} 7 \\ 4 \\ 5 \end{pmatrix}$	3	1	3×1
3	$\begin{pmatrix} 7 & 8 \\ 2 & 4 \end{pmatrix}$	2	2	2×2
4	$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$	3	3	3×3
5	$\begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}$	2	3	2×3
6	$\begin{pmatrix} 1 & 2 & 3 & 5 & 7 \\ 4 & 5 & 6 & 0 & 1 \\ 7 & 8 & 9 & 2 & 1 \\ 0 & 5 & 1 & 8 & 6 \end{pmatrix}$	4	5	4×5

Note that if a matrix has only 1 row, then it is called a row matrix. Similarly if it has only 1 column, then it is called a column matrix.

5.1.2 Types of Matrices

The following are the different types of matrices

1. **Row Matrix:** This is a matrix with only 1 row, for example

$$\begin{pmatrix} 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 5 & 0 \end{pmatrix}, \begin{pmatrix} 5 & 0 & 9 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 4 \end{pmatrix}$$

2. **Column Matrix:** This is a matrix with only 1 column, for example

$$\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 9 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

3. **Zero Matrix:** This is a matrix in which all the entries are 0, for example

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

4. **Square Matrix:** This is a matrix in which its number of rows is equal to its number columns, for example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \\ 5 & 6 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 1 & 2 & 5 \\ 3 & 4 & 3 & 4 & 1 \\ 5 & 6 & 1 & 1 & 6 \\ 2 & 3 & 4 & 5 & 9 \\ 1 & 4 & 6 & 6 & 1 \end{pmatrix}$$

5. **Diagonal Matrix:** This is a square matrix in which all its entries are 0 except the leading diagonal entries, for example

$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}$$

6. **Identity Matrix:** This is a diagonal matrix in which all its leading diagonal entries are 1, for example

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

7. **Upper Triangular Matrix:** This is a square matrix in which all entries below its leading diagonal are 0, for example

$$\begin{pmatrix} 8 & 2 \\ 0 & -4 \end{pmatrix}, \begin{pmatrix} 5 & 4 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 9 \end{pmatrix}, \begin{pmatrix} 6 & 4 & 10 & 25 \\ 0 & 2 & 3 & 11 \\ 0 & 0 & 7 & -2 \\ 0 & 0 & 0 & 17 \end{pmatrix}, \begin{pmatrix} 4 & -1 & 3 & 5 & 3 \\ 0 & 5 & -4 & 3 & 4 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}$$

8. **Lower Triangular Matrix:** This is a square matrix in which all entries above its leading diagonal are 0, for example

$$\begin{pmatrix} 3 & 0 \\ 4 & 6 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 5 & 4 & 0 \\ 2 & 1 & 8 \end{pmatrix}, \begin{pmatrix} -9 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 2 & 4 & 8 & 0 \\ 2 & 5 & 7 & 7 \end{pmatrix}, \begin{pmatrix} 8 & 0 & 0 & 0 & 0 \\ 9 & 6 & 0 & 0 & 0 \\ 5 & 6 & 15 & 0 & 0 \\ 4 & 3 & 5 & 7 & 0 \\ 4 & 6 & 7 & 77 & 4 \end{pmatrix}$$

9. **Transpose of a Matrix:** The transpose of a matrix A written as A^T is obtained by changing all its rows to columns, for example if

$$A = \begin{pmatrix} 3 & 8 \\ 5 & 6 \end{pmatrix}$$

then

$$A^T = \begin{pmatrix} 3 & 5 \\ 8 & 6 \end{pmatrix}$$

also if

$$B = \begin{pmatrix} 1 & 3 & 8 \\ 2 & 5 & 6 \\ 7 & 9 & 2 \end{pmatrix}$$

then

$$B^T = \begin{pmatrix} 1 & 2 & 7 \\ 3 & 5 & 9 \\ 8 & 6 & 2 \end{pmatrix}$$

10. **Symmetric Matrix:** This is a square matrix which remains unchanged when its transpose is taken, for example

$$\begin{pmatrix} 4 & -2 \\ -2 & 9 \end{pmatrix}, \begin{pmatrix} 2 & 7 & 3 \\ 7 & 9 & 4 \\ 3 & 4 & 7 \end{pmatrix},$$

11. **Skew - symmetric Matrix:** This is a square matrix A which $A^T = -A$, i.e. a matrix whose transpose is equal to the matrix obtained by multiplying each entry of the matrix by -1, for example

$$\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & -5 \\ -2 & 5 & 0 \end{pmatrix}$$

5.2 Addition/Subtraction of Matrices

Only matrices with the same order or dimension can be added/s subtracted and the addition/subtraction is done entry by entry, for example

Examples: Evaluate the following

1. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

   Solution:

   $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

2. $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 6 \\ 7 & 8 & 9 \end{pmatrix} - \begin{pmatrix} 5 & 6 & 1 \\ 1 & 7 & 8 \\ 5 & 2 & 0 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 2 & -3 & 6 \\ 3 & 4 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 6 & \frac{1}{2} & 6 \\ 1 & 6 & 7 & 8 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 6 \\ 7 & 8 \\ -1 & 0 \\ 2 & -3 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 2 & 2 \\ 3 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}$

5.3 Multiplication of Matrices

Multiplication of matrices can be done in 2 ways, the scalar multiplication and the matrix multiplication. These are explained below

5.3.1 Scalar Multiplication

This is when a matrix is multiplied by a number. This is done by multiplying the number by each entry of the matrix.

Examples: Multiply the following matrices by the scalars

1. $5 \times \begin{pmatrix} 1 & 2 & 3 & 6 \\ 3 & 4 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 5\times 1 & 5\times 2 & 5\times 3 & 5\times 6 \\ 5\times 3 & 5\times 4 & 5\times 1 & 5\times 0 \end{pmatrix} = \begin{pmatrix} 5 & 10 & 15 & 30 \\ 15 & 20 & 5 & 0 \end{pmatrix}$

Solution:

2. If $A = \begin{pmatrix} 5 & 1 & 0 \\ 4 & 1 & 1 \\ 8 & 6 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 8 & 2 \\ 0 & 5 & 2 \\ 7 & 2 & 1 \end{pmatrix}$, find $5A - 2B$.

3. $\frac{2}{3} \times \begin{pmatrix} 12 & 80 \\ 42 & 44 \\ 26 & 62 \end{pmatrix}$

5.3.2 Matrix Multiplication

Multiplication of matrices is done in a special way using a special procedure. Multiplication of matrices is NOT DONE ENTRY BY ENTRY. The procedure of multiplying matrices is illustrated below using the example below

Supposing you are asked to multiply

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ by $\begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}$

The following steps should be followed

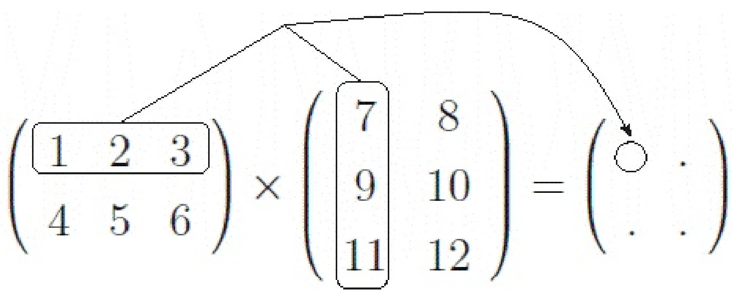
Step 1: Check whether the 2 matrices are compatible because only compatible matrices can be multiplied. Two matrices are compatible for matrix multiplication if the the number of rows of the first matrix is equal to the number of columns of the second matrix. So looking at the example we can see that

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}$$

The number of rows of the first matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is 2 and the number of columns of the second matrix $\begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}$ is also 2, therefore the two matrices are compatible and we can go ahead and multiply.

Note that if 2 matrices are not compatible, then they can not be multiplied. Since the two matrices in our example are compatible, then we proceed to the next step.

Step 2: Multiply element of the first row in the first matrix by elements of the first column in the second matrix and add the results to obtain the first entry of the answer, see figure below



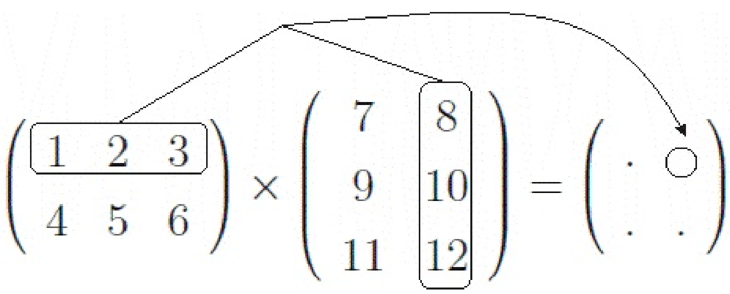
When multiplying, you start from the first element in the first row of the first matrix, then multiply it by the first element in the first column of the second matrix followed by the second element of the row multiplied by the second element of the column and the followed by the third element of the row multiplied by the third element of the column afterwards, you add everything up, thus

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \times \begin{pmatrix} 7 \\ 9 \\ 11 \end{pmatrix} = (1\times7)+(2\times9)+(3\times11) = 7+18+33 = 58$$

we have now gotten the first entry of our answer, therefore we have

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 58 & . \\ . & . \end{pmatrix}$$

Step 3: We then multiply the first row of the first matrix by the second column of the second matrix in the same manner and get the second entry of the first row our answer, i.e.



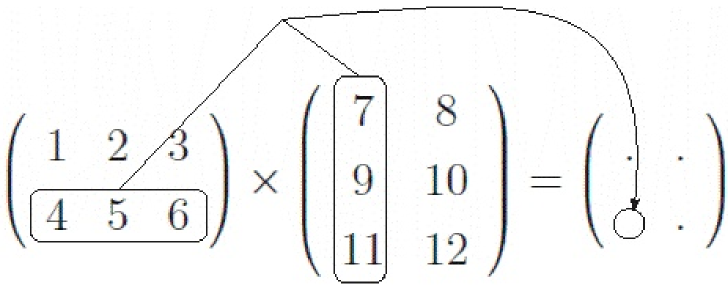
Multiplying we get

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \times \begin{pmatrix} 8 \\ 10 \\ 12 \end{pmatrix} = (1\times8)+(2\times10)+(3\times12) = 8+20+36 = 64$$

therefore we have

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 58 & 64 \\ . & . \end{pmatrix}$$

Step 4: We then multiply the second row of the first matrix by the first column of the second matrix in the same manner and get the first entry of the second row of our answer, i.e.



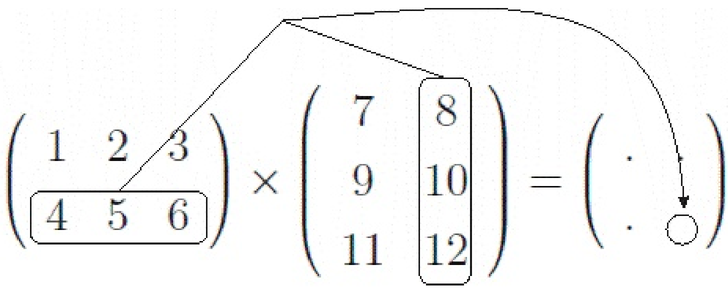
Multiplying we get

$$\begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 \\ 9 \\ 11 \end{pmatrix} = (4\times7)+(5\times9)+(6\times11) = 28+45+66 = 139$$

therefore we have

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 58 & 64 \\ 139 & . \end{pmatrix}$$

Step 5: We then multiply the second row of the first matrix by the second column of the second matrix in the same manner and get the second entry of the second row of our answer, i.e.



Multiplying we get

$$\begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 8 \\ 10 \\ 12 \end{pmatrix} = (4\times8)+(5\times10)+(6\times12) = 32+50+72 = 154$$

therefore we have

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 58 & 64 \\ 139 & 154 \end{pmatrix}$$

The following laws can be proven for matrix addition

1. Cummutative Law :$A + B = B + A$

   Proof:

   Let
   $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$
   and
   $$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{pmatrix}$$
   then

$$
\begin{aligned}
A+B &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} & \cdots & a_{2n}+b_{2n} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} & \cdots & a_{3n}+b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & a_{m3}+b_{m3} & \cdots & a_{mn}+b_{mn} \end{pmatrix} \\
&= \begin{pmatrix} b_{11}+a_{11} & b_{12}+a_{12} & b_{13}+a_{13} & \cdots & b_{1n}+a_{1n} \\ b_{21}+a_{21} & b_{22}+a_{22} & b_{23}+a_{23} & \cdots & b_{2n}+a_{2n} \\ b_{31}+a_{31} & b_{32}+a_{32} & b_{33}+a_{33} & \cdots & b_{3n}+a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{m1}+a_{m1} & b_{m2}+a_{m2} & b_{m3}+a_{m3} & \cdots & b_{mn}+a_{mn} \end{pmatrix} \\
&= \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \\
&= B+A
\end{aligned}
$$

2. Associative Law: $(A+B)+C = A+(B+C)$

3.

Examples:

1. **Solution:**

2. **Solution:**

Examples:

1. **Solution:**

2. **Solution:**